Uniform amenability and hyperfiniteness of treeable equivalence relations

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- E is **hyperfinite** if $E = \bigcup_{n=1}^{\infty} E_n$ where $E_n \subseteq E_{n+1}$ and each E_n is a finite CBER.
- E is **treeable** if there is an acyclic Borel graph $\mathcal{G} = (X, R)$ such that $E = E_{\mathcal{G}}$.

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A tree with a cycle.

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Hyperfinite CBERs are moreover **(Frechet-)amenable**, as defined by Jackson–Kechris–Louveau. Whether the vice versa holds is among the most challenging open questions in the subject.

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Connes–Feldman–Weiss, 1981: If G is countable **amenable** and $G \curvearrowright X$ is Borel, then E^X_G is μ-**hyperfinite**, $\mu \in M(X)$.

Weiss, 1984 Suppose G is countable, amenable and $G \curvearrowright X$ is Borel. Is E^X_G hyperfinite?

The answer is known to be positive for many classes of groups.

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Marquis–Sabok, 2020: $E_G^{\partial G}$ is hyperfinite, for the boundary action $G \curvearrowright \partial G$ of any finitely generated **hyperbolic group**.

The latter result has been expanded in several directions (e.g. Karpinski, Naryshkin–V., Oyakawa).

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Jackson–Kechris–Louveau, 2002: Are amenable treeable CBERs hyperfinite?

The main results

Theorem (Naryshkin–V., 2025)

Let $\mathcal{G} = (X, R)$ be an acyclic Borel graph with bounded degree. If $E_{\mathcal{G}}$ is ***uniformly amenable with respect to*** $\rho_{\mathcal{G}}$*, then $\mathrm{asdim}_B(X, \rho_{\mathcal{G}}) < \infty$, and in particular $E_{\mathcal{G}}$ is hyperfinite.*

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Corollary (Naryshkin–V., 2025)

Let $F_k \curvearrowright X$ be a continuous, free, amenable action on a σ-compact Polish space. Then $E^X_{F_k}$ is hyperfinite.

Corollary (Naryshkin–V., 2025)

Let G be a countable amenable group and $G \curvearrowright X$ a Borel action. If E^X_G is treeable, then it is hyperfinite.

Some notation

A **Borel extended metric space** (X, ρ) is a standard Borel set with a Borel metric ρ that can also have value ∞.

$$E_\rho := \{(x, y) \in X^2 : \rho(x, y) < \infty\}.$$

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Example. A Borel graph $\mathcal{G} = (X, R)$ with the shortest path metric $\rho_\mathcal{G}$. In this case $E_{\rho_\mathcal{G}} = E_\mathcal{G}$.

If G is a finitely generated group with finite symmetric set of generators S, and $G \curvearrowright X$ is a Borel action, the **Schreier graph** $\mathcal{G} = (X, R)$ is defined as

$$xRy \iff \exists g \in S \setminus \{e\} \text{ such that } gx = y.$$

Borel asymptotic dimension

Definition (Conley–Jackson–Marks–Seward–Tucker-Drob, 2023)

Let (X, ρ) be an extended metric space. The **Borel asymptotic dimension** of (X, ρ), denoted $\mathrm{asdim}_B(X, \rho)$, is the smallest $d \in \mathbb{N}$ such that for every $r > 0$ there is a ρ-uniformly bounded Borel equivalence relation E such that $B_\rho(x, r)$ meets at most $d + 1$ E-classes, and it is ∞ if no such d exists.

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Theorem (Conley–Jackson–Marks–Seward–Tucker-Drob, 2023)

If (X, ρ) is proper and $\mathrm{asdim}_B(X, \rho) < \infty$ then E_ρ is hyperfinite.

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If $(\rho_n)_{n=1}^\infty$ are proper Borel extended metrics on X such that $\rho_n \leq \rho_{n+1}$, $E_\rho = \bigcup_{n=1}^\infty E_{\rho_n}$ and $\mathrm{asdim}_B(X, \rho_n) < \infty$, then E_ρ is hyperfinite.

Uniform amenability

Definition (Naryshkin–V., 2025)

Let (X, ρ) be a Borel extended metric space and let E be a CBER. E is **uniformly amenable with respect to ρ** if $E \subseteq E_\rho$ and if there are Borel maps

$$\lambda_n \colon E \to [0,1], \quad n \in \mathbb{N},$$

such that, with $\lambda_{n,x}(\cdot) := \lambda_n(x, \cdot)$,

- $\lambda_{n,x} \in \ell^1([x]_E)$ and $\|\lambda_{n,x}\|_1 = 1$, for every $x \in X$
- $\sup_{\{(x,y)\in E : \rho(x,y)<r\}} \|\lambda_{n,x} - \lambda_{n,y}\|_1 \to 0$, for every $r > 0$.

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- $\sup_{\{(x,y)\in E : \rho(x,y)<r\}} \|\lambda_{n,x} - \lambda_{n,y}\|_1 \to 0$, for every $r > 0$.

Uniform amenability differs from **amenability** because of (2), where the latter only requires

$$\|\lambda_{n,x} - \lambda_{n,y}\|_1 \to 0, \quad \text{for all } xEy.$$

Uniformly amenable CBERs

Proposition

Let G be a finitely generated amenable group, let $G \curvearrowright X$ be a Borel action and let $\mathcal{G}$ be the Schreier graph generated by a finite symmetric set of generators of G. Then E_G^X is uniformly amenable with respect to $\rho_{\mathcal{G}}$.

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Poof. Let $(F_n)_{n=1}^{\infty}$ be a Følner sequence, and set

$$\begin{aligned} \lambda_n \colon E_{\mathcal{G}} &\to [0,1] \\ (x,y) &\mapsto \frac{1}{|F_n|}|\{g \in G_n : gx = y\}| \end{aligned}$$

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Proposition

Let G be a finitely generated group G, X a compact Polish group and $G \curvearrowright X$ a continuous, amenable action. If $\mathcal{G}$ is as above, then E_G^X is uniformly amenable with respect to $\rho_{\mathcal{G}}$.

The main results (again)

Theorem (Naryshkin–V., 2025)

Let $\mathcal{G} = (X, R)$ be an acyclic Borel graph with bounded degree. If $E_{\mathcal{G}}$ is ***uniformly amenable with respect to $\rho_{\mathcal{G}}$****, then $\mathrm{asdim}_B(X, \rho_{\mathcal{G}}) < \infty$, and in particular $E_{\mathcal{G}}$ is hyperfinite.*

Corollary (Naryshkin–V., 2025)

Let $F_k \curvearrowright X$ be a continuous free action on a σ-compact Polish space. If $E_{F_k}^X$ is amenable, then it is hyperfinite.

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Let G be a countable amenable group and $G \curvearrowright X$ a Borel action. If E_G^X is treeable, then it is hyperfinite.

Borel orientations

Let $\mathcal{G} = (X, R)$ be a Borel graph. A **Borel orientation** is a Borel subset $\overrightarrow{R} \subseteq R$ such that $\overrightarrow{R} \cap \overrightarrow{R}^{-1} = \emptyset$ and $\overrightarrow{R} \cup \overrightarrow{R}^{-1} = R$.

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Let $\mathcal{G} = (X, R)$ be a Borel graph with bounded degree. If $\mathcal{G}$ has a Borel orientation with out-degree at most 1, then $\mathrm{asdim}_B(X, \rho_{\mathcal{G}}) \leq 1$.

Proposition (Conley–Jackson–Marks–Seward–Tucker-Drob)

Suppose that $f : X \to X$ is Borel and bounded-to-one, and $\mathcal{G}_f = (X, R_f)$ where $x R_f y$ iff $f(x) = y$ or $f(y) = x$. Then $asdim_B(X, \rho_{\mathcal{G}_f}) \leq 1$.

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Proof. $\mathcal{G} = \mathcal{G}_f$ where f is the following bounded-to-one Borel function

$$f(x) := \begin{cases} y & \text{if } x\overrightarrow{R}y \\ x & \text{otherwise} \end{cases}$$

Partial orientations

Proposition

Let $\mathcal{G} = (X, R)$ be a Borel graph with bounded degree. Suppose that for every $r > 0$ there is a Borel symmmetric $Q \subseteq R$ such that

- $\rho_{\mathcal{G}}(q_0, q_1) \geq r$ *for all* $q_0, q_1 \in Q$ *distinct,*
- $(X, R \setminus Q)$ *has a Borel orientation with out-degree at most 1.*

Then $\mathrm{asdim}_B(X, \rho_{\mathcal{G}}) \leq 3$.

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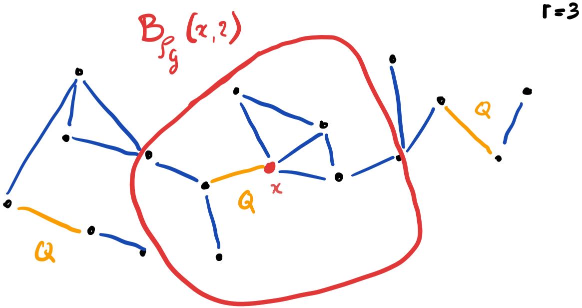

Let $\mathcal{G} = (X, R)$ be an acylcic Borel graph of bounded degree and such that $E_{\mathcal{G}}$ is uniformly amenable with respect to $\rho_{\mathcal{G}}$. Fix $r > 0$, and find a Borel map

$$\lambda\colon E_{\mathcal{G}} \to [0,1],$$

such that

- $\lambda_x \in \ell^1([x]_{\mathcal{G}})$, $\|\lambda_x\|_1 = 1$, for every $x \in X$,
- $\|\lambda_x - \lambda_y\|_1 < 1/12$ if $\rho_{\mathcal{G}}(x, y) < r + 2$.

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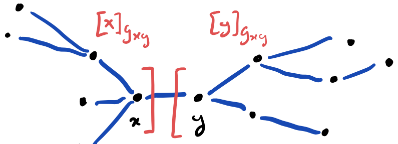
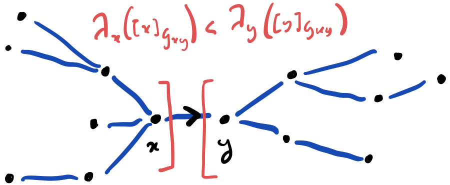
Take xRy.



Let $\mathcal{G}_{xy} = (X, R \setminus \{(x,y),(y,x)\})$. Then
$\lambda_x([x]_{\mathcal{G}_{xy}}) + \lambda_x([y]_{\mathcal{G}_{xy}}) = 1$ and $\lambda_y([x]_{\mathcal{G}_{xy}}) + \lambda_y([y]_{\mathcal{G}_{xy}}) = 1$

Let $R_0 := \{(x, y) \in R : \lambda_x([x]_{\mathcal{G}_{xy}}), \lambda_y([y]_{\mathcal{G}_{xy}}) \notin [5/12, 7/12]\}$ and orient $x \vec{R}_0 y$ only if $\lambda_x([x]_{\mathcal{G}_{xy}}) < \lambda_y([y]_{\mathcal{G}_{xy}})$.



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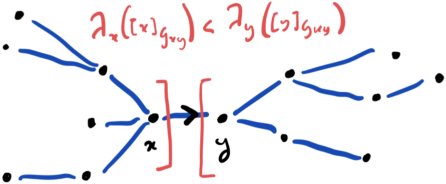
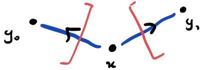


This has a number of consequences:

$\vec{R}_0$ has out-degree ≤ 1.



$\lambda_x([y_0]_{\mathcal{G}_{xy_0}}) + \lambda_x([y_1]_{\mathcal{G}_{xy_1}}) > 1$

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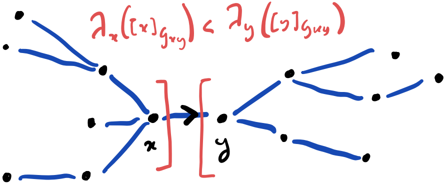

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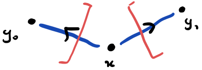
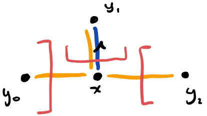


$\lambda_x([y_0]_{\mathcal{G}_{xy_0}}) + \lambda_x([y_1]_{\mathcal{G}_{xy_1}}) > 1$

Set $R_1 := R \setminus R_0$. Then $\deg_{R_1}(x) + \deg^{\text{out}}_{\vec{R}_0}(x) \leq 2$.



$\lambda_x([y_0]_{\mathcal{G}_{xy_0}}) + \lambda_x([y_1]_{\mathcal{G}_{xy_1}}) + \lambda_x([y_2]_{\mathcal{G}_{xy_2}}) > 1$

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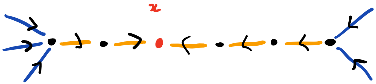

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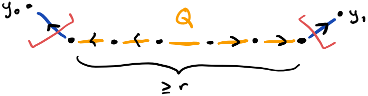
Case 3. Both leaves in $[x]_{\mathcal{G}_1}$ have out-degree 1.



because of measure restrictions.

Summarizing:

- use the measure given by uniform amenability to define a partial orientation.
- enlarge the orientation so that the left-over, non-oriented part Q is composed by a Borel set of sparse edges that make all components of $\mathcal{G}_1$ finite, plus the 'central' edges in connected components with two leaves with out-degree 1.

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Thank you!